NUMERICAL SOLUTION OF A PARABOLIC EQUATION

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A splitting scheme is proposed for the numerical solution of a parabolic equation containing singularities important on the boundary of the region of the sought function.

In the rectangular region G = {0 < x < a, 0 < y < b} for t > 0 it is required to find the function u(x, y, t) satisfying the equation

$$\frac{\partial u}{\partial t} - \Lambda u \equiv \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial y^2} + K(x, y)\frac{\partial u}{\partial x} + d(x, y)u = f(x, y, t) \tag{1}$$

(where the bounded function K(x, y) can change sign), with initial and boundary conditions

$$u|_{t=0} = u_0(x, y),\ x,\ y \in G, \tag{2}$$

$$u|_\Gamma = g(x, y, t), \tag{3}$$

where Γ is part of the boundary of the region G formed by the straight-line segments

$$\left.\begin{matrix} 0 < x < a,\ y = 0, \\ 0 < x < a,\ y = b \end{matrix}\right\} \tag{4}$$

and the parts of the straight-line segments

$$\left.\begin{matrix} 0 < y < b,\ x = 0 \ \text{ for } K(0, y) > 0, \\ 0 < y < b,\ x = a \ \text{ for } K(a, y) < 0 \end{matrix}\right\} \tag{5}$$

(see Fig. 1).

This problem will be a correctly formulated problem of Fikera [1] if d(x, y) and K_x(x, y) are bounded.

The exact solution of problem (1)-(3) is not smooth and has singularities at those points where the coefficient K(x, y) changes sign. It is assumed that the exact solution can be well approximated by a smooth function for appropriate functional spaces for u, f, u_0, g, so that approximation of the derivatives by finite-difference expressions would become possible.

We assume the following splitting scheme for solution of the problem. We represent the operator Λ in (1) in the form of the sum of two operators of constant sign K(x, y) + C, taking the constant C > max{K(x, y)}:

$$\Lambda u = \Lambda_1 u + \Lambda_2 u,$$

where

$$\left.\begin{matrix} \Lambda_1 u = -\frac{1}{2}\frac{\partial^2 u}{\partial y^2} + (K(x, y) + C)\frac{\partial u}{\partial x} + d(x, y)u, \\ \Lambda_2 u = -\frac{1}{2}\frac{\partial^2 u}{\partial x^2} - C\frac{\partial u}{\partial x}. \end{matrix}\right\} \tag{6}$$

It is necessary to give the boundary conditions at x = 0 for Λ_1 and at x = a for Λ_2. Realizing the method of fractional steps

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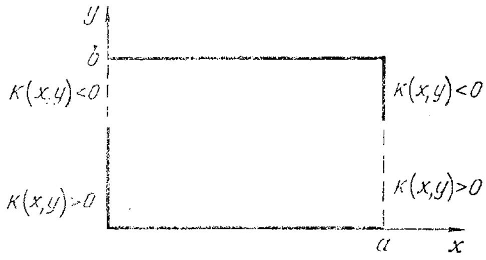

Fig. 1. Boundary of the region of variation of the function u(x, y, t).

$$\left.\begin{aligned}\frac{u\left(t+\frac{\tau}{2}\right)-u(t)}{\tau/2} &= \Lambda_1 u\left(t+\frac{\tau}{2}\right)+\frac{1}{2}f\left(t+\frac{\tau}{2}\right),\\ \frac{u(t+\tau)-u\left(t+\frac{\tau}{2}\right)}{\tau/2} &= \Lambda_2 u(t+\tau)+\frac{1}{2}f(t+\tau),\end{aligned}\right\} \tag{7}$$

it follows that in order to find $u\left(t+\frac{\tau}{2}\right)$ we must assign values at x = 0 according to (5) if K(0, y) > 0, and when K(0, y) $\leqslant$ 0 we must take values of u(0, y, t) calculated from the preceding step (or from (2) at t = 0). To find u(t + τ) we must assign values at x = a according to (5) if K(a, y) < 0, and when K(a, y) $\geqslant$ 0 we must use the values of $u\left(t+\frac{\tau}{2}\right)$ found at (a, y).

Thus, the boundary conditions for the calculation are taken partly from the assigned boundary conditions and partly from the preceding time step, which makes the splitting (7) not entirely standard.

Solution of system (7), written in a finite-difference form, was carried out by the pivotal method according to an implicit scheme. We note that owing to the singularities of the exact solution of problem (1)-(3), the advantage of applying the more exact traditional schemes will be insignificant if the approximation is not adjusted to eliminate the specific singularities.

The procedure discussed here was used for calculating the temperature field in a plane channel with vibrating sides [2].

LITERATURE CITED

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